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Numerical Studies for Solving a Free Convection Boundary–Layer Flow Over a Vertical Plate

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In this paper, The aim of this study is to present a reliable combination of the shifted Legendre collocation method to approximate of the problem of free convection boundary-layer flow over a vertical plate as produced by a body force about a flat plate in the direction of the generating body force. The proposed method is based on replacement of the unknown function by truncated series of well known shifted Legendre expansion of functions. An approximate formula of the integer derivative is introduced. Special attention is given to study the convergence analysis and derive an upper bound of the error of the presented approximate formula. The introduced method converts the proposed equation by means of collocation points to a system of algebraic equations with shift Legendre coefficients. Thus, by solving this system of equations, the shifted Legendre coefficients are obtained. Boundary conditions in an unbounded domain, i.e. boundary condition at infinity, pose a problem in general for the numerical solution methods. The obtained results are in good agreement with those provided previously by the iterative numerical method. As a result, without taking or estimating missing boundary conditions, the shifted Legendre collocation method provides a simple, non-iterative and effective way for determining the solutions of nonlinear free convection boundary layer problems possessing the boundary conditions at infinity.

Keywords: nonlinear differential equations, free convection boundary-layer flow, heat transfer, shifted Legendre collocation method.

1. Introduction

A great number of the nonlinear phenomena can be modeled by nonlinear differential equations in many areas of scientific fields such as engineering, physics, biology, fluid mechanics. Since convection problems come across both in nature and engineering applications, they have attracted a great deal of attention from researchers. Free convection flow problems, which results from the action of body forces on the fluid, are one of the common areas of interest in the field of convection problems. For the sake of simplicity, the many of free convection boundary-layer problems are considered as a special case of free convection flow about a flat plate parallel to the direction of the generating body force. The most notable model of this topic is the experimental and theoretical considerations of Schmidt and Beckmann [1] concerning the free convection flow of air subject to the gravitational force about a vertical flat plate [2].

This type of free convection boundary-layer problem was studied at the NACA Lewis Laboratory during 1951 and then was analyzed by an iterative numerical method by Ostrach in 1953. After that Na and Habib [3] solved these problems by a parameter method in 1974. In 2005, Kuo [4] employed these problems with the differential trans-formation method (DTM). In [3] and [4], the boundary conditions of $f^{''}(0)$ and $\theta^{'}(0)$ is taken from the Ostrach [2].

The nonlinear differential equations which have boundary conditions in unbounded domains have a great interest. However, many of the modeled nonlinear equations do not have an analytical solution. Both analytical solution methods and numerical solution methods are used to solve these equations. In this study, the shifted Legendre is one of the effective and reliable numerical solution method for handling both linear and nonlinear differential equations. In order to overcome the difficulty in unbounded domains, i.e. infinity boundary conditions are widely used.

The present study has an important due to the fact that this problem is a heat transfer problem consisting boundary conditions at infinity and is solved by the Shifted Legendre collocation method. In addition, instead of discrete solutions, continuous solutions are obtained without taking initial conditions and/or estimations for the lack of boundary conditions.

Legendre polynomials are well known family of orthogonal polynomials on the interval [1, -1] that have many applications. They are widely used because of their good properties in the approximation of functions [5-8]. Orthogonal polynomials have a great variety and wealth of properties. Some of these properties take a very concise form in the case of the Legendre polynomials, making Legendre polynomials of leading importance among orthogonal polynomials. The Legendre polynomials belong to an exclusive band of orthogonal polynomials, known as Jacobi polynomials, which correspond to weight functions of the form $(1-x)^{\alpha}(1+x)^{\beta}$ and which are solutions of Sturm-Liouville equations [5]. The Legendre collocation method is used to solve many problems, in more papers such as [5-8]. In this work, we use the properties of the Legendre polynomials to derive an approximate formula of the integer derivative $D^{(n)}y(x)$ and estimate an error upper bound of this formula, then we use this formula to solve numerically the proposed problem.

2. Mathematical analysis

We consider laminar free-convection flow of an incompressible viscous fluid about a flat plate parallel to the direction of the generating body force. The physical model is shown in Fig.1. By the assumption that the flow in the laminar boundary layer is two dimensional, the continuity equation, the momentum equation, the energy equation and the boundary conditions can be expressed as [1]:

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = 0 \tag{1}$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = g\beta(T - T_\infty) + v\frac{\partial^2 u}{\partial y^2} \tag{2}$$

$$u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = \alpha\frac{\partial^2 T}{\partial y^2} \tag{3}$$

With the boundary conditions:

$$\text{at} \quad x = 0 \quad u = 0 \quad \text{and} \quad T = T_\infty \tag{4}$$

$$\text{at} \quad y = 0 \quad u = v = 0 \quad \text{and} \quad T = T_0 \tag{5}$$

$$\text{as} \quad y \to \infty \quad u = 0 \quad \text{and} \quad T = T_\infty \tag{6}$$

where u and v are the velocity in the x and y direction respectively, ν is the viscosity of the fluid, α is the thermal diffusivity of the fluid.

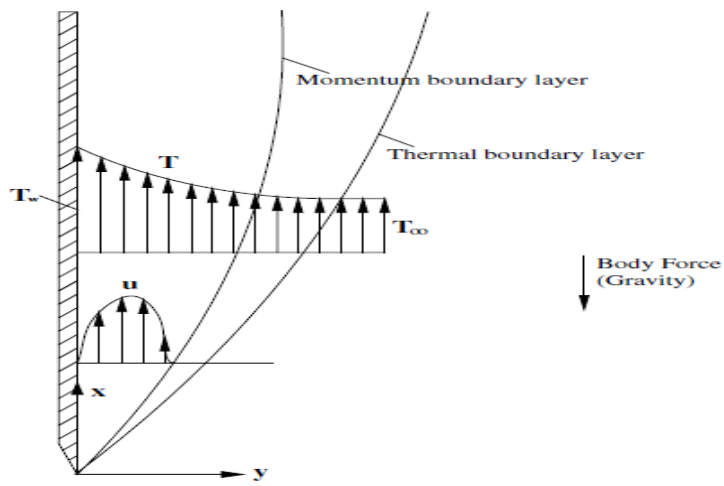

Figure 1 Free convection flow over a vertical plate

After a group of transformations [2], the equations (1–3) with respect to the boundary conditions (4–6) reduce to the following form:

$$f^{'''}(\eta) + 3f(\eta)f^{''}(\eta) - 2[f^{'}(\eta)]^2 + \theta = 0 \tag{7}$$

$$\theta^{''}(\eta) + 3P_r f(\eta)\theta^{'}(\eta) = 0 \tag{8}$$

The boundary conditions for the equations (7-8) are as follows:

$$\text{at} \quad \eta = 0 \quad f = f^{'} = 0 \quad \theta = 1\text{as} \quad \eta \to \infty \quad f^{'} = 0 \quad \theta = 0 \tag{9}$$

where P_r is the Prandtl number.

3. An approximate formula of the integer derivative for Legendre polynomial expansion

The well known Legendre polynomials are defined on the interval $[-1, 1]$ and can be determined with the aid of the following recurrence formula [2]:

$$L_{k+1}(z) = \frac{(2k+1)}{(k+1)} z L_k(z) - \frac{k}{k+1} L_{k-1}(z), \quad k = 1,\, 2, \ldots$$

where, $L_0(z) = 1$ and $L_1(z) = z$. In order to use these polynomials on the interval $[0, 1]$ we define the so called shifted Legendre polynomials by introducing the change of variable $z = 2t - 1$.

Let the shifted Legendre polynomials $L_k(2t - 1)$ be denoted by $P_k(t)$. Then $P_k(t)$ can be obtained as follows:

$$P_{k+1}(t) = \frac{(2k+1)(2t-1)}{(k+1)} P_k(t) - \frac{k}{k+1} P_{k-1}(t), \quad k = 1, 2, \ldots \tag{10}$$

where, $P_0(t) = 1$ and $P_1(t) = 2t - 1$. The analytic form of the shifted Legendre polynomial $P_k(x)$ of degree k is given by:

$$P_k(t) = \sum_{i=0}^{k} (-1)^{k+i} \frac{(k+i)!}{(k-i)(i!)^2} t^i \tag{11}$$

Note that $P_k(0) = (-1)^k$ and $P_k(1) = 1$. The orthogonality condition is:

$$\int_0^1 P_i(t) P_j(t)\, dx = \begin{cases} \frac{1}{2i+1} & \text{for} \quad i = j \\ 0 & \text{for} \quad i \neq j \end{cases} \tag{12}$$

The function $y(t)$, which is square integrable in $[0, 1]$, may be expressed in terms of shifting Legendre polynomial as:

$$y(t) = \sum_{i=0}^{\infty} y_i P_i(t)$$

where the coefficients y_i are given by $y_i = (2i + 1) \int_0^1 y(t) P_i(t)\, dt,\ i = 1, 2, \ldots$.
In practice, only the first $(m+1)$-terms shifted Legendre polynomials are considered. Then we have:

$$y_m(t) = \sum_{i=0}^{m} y_i P_i(t). \tag{13}$$

In the following theorem we will approximate the fractional derivative of $y(t)$.

Theorem 1. [6-8]:

Let $y(t)$ be approximated by shifting Legendre polynomials as (13) then the integer derivative of order n is given by:

$$y_m^{(n)}(t) = \sum_{i=n}^{m} \sum_{k=n}^{i} y_i w_{i,k}^{(n)} t^{k-n} \tag{14}$$

Where, $w_{i,k}^{(n)}$ is given by:

$$w_{i,k}^{(n)} = \frac{(-1)^{(i+k)}(i+k)!}{(i-k)!(k)!\Gamma(k+1-n)} \tag{15}$$

Proof:

Since the differentiation is a linear operation, then from (13) we get:

$$D^{(n)}(y_m(t)) = \sum_{i=0}^{m} y_i D^{(n)}(P_i(t)) \tag{16}$$

From the formula (10) we can obtain:

$$D^{(n)} P_i(t) = 0, \qquad i = 0, 1, ..., n-1. \tag{17}$$

Therefore, for $i = n,\ n+1, ...,\ m$, and with (10) we get:

$$D^{(n)} P_i(t) = \sum_{k=0}^{i} \frac{(-1)^{i+k}(i+k)!}{(i-k)!(k!)^2} D^{(n)}(t^k) = \sum_{k=n}^{i} \frac{(-1)^{i+k}(i+k)!}{(i-k)!(k!)\Gamma(k+1-n)} t^{k-n} \tag{18}$$

a combination of Eqs. (16–18) leads to the desired result.

Theorem 2. Legendre truncation theorem [5]:

The error in approximating $x(t)$ by the sum of its first m terms is bounded by the sum of the absolute values of all the neglected coefficients. If:

$$y_m(t) = \sum_{k=0}^{m} c_k L_k(t) \tag{19}$$

then:

$$E_T(m) \equiv |y(t) - y_m(t)| \leq \sum_{k=m+1}^{\infty} |c_k| \tag{20}$$

for all $y(t)$, m and $t \in [-1, 1]$.

4. Procedure solution

In this section, we present the proposed method to solve numerically the system of ordinary differential equations of the form (7–8). The unknown functions $f(\eta)$ and $\theta(\eta)$ may be expanded by finite series of shifting Legendre polynomials as the following approximation:

$$f_m(\eta) = \sum_{i=0}^{m} c_i L^*(\eta) \qquad \theta_m(\eta) = \sum_{i=0}^{m} d_i L^*(\eta) \tag{21}$$

From Eqs. (7–8), (21) and Theorem 1 we have:

$$\sum_{i=3}^{m}\sum_{k=3}^{i} c_i\gamma_{i,k}^{(3)}\eta^{k-3} + 3\left(\sum_{i=0}^{m} c_i L_i^*(\eta)\right)\left(\sum_{i=2}^{m}\sum_{k=2}^{i} c_i\gamma_{i,k}^{(2)}\eta^{k-2}\right)$$
$$-2\left(\sum_{i=1}^{m}\sum_{k=1}^{i} c_i\gamma_{i,k}^{(1)}\eta^{k-1}\right) + \sum_{i=0}^{m} d_i L_i^*(\eta) = 0 \quad (22)$$

$$\sum_{i=2}^{m}\sum_{k=2}^{i} d_i\gamma_{i,k}^{(2)}\eta^{k-2} + 3P_r\left(\sum_{i=0}^{m} c_i L_i^*(\eta)\right)\left(\sum_{i=1}^{m}\sum_{k=1}^{i} d_i\gamma_{i,k}^{(1)}\eta^{k-1}\right) = 0 \quad (23)$$

We now collocate Eqs. (22–23) at $(m-n+1)$ points $\eta_s,\ \ s = 0,\ 1, ...,\ m-n$ as:

$$\sum_{i=3}^{m}\sum_{k=3}^{i} c_i\gamma_{i,k}^{(3)}\eta_s^{k-3} + 3\left(\sum_{i=0}^{m} c_i L_i^*(\eta_s)\right)\left(\sum_{i=2}^{m}\sum_{k=2}^{i} c_i\gamma_{i,k}^{(2)}\eta_s^{k-2}\right)$$
$$-2\left(\sum_{i=1}^{m}\sum_{k=1}^{i} c_i\gamma_{i,k}^{(1)}\eta_s^{k-1}\right) + \sum_{i=0}^{m} d_i L_i^*(\eta_s) = 0 \quad (24)$$

$$\sum_{i=2}^{m}\sum_{k=2}^{i} d_i\gamma_{i,k}^{(2)}\eta_s^{k-2} + 3P_r\left(\sum_{i=0}^{m} c_i L_i^*(\eta_s)\right)\left(\sum_{i=1}^{m}\sum_{k=1}^{i} d_i\gamma_{i,k}^{(1)}\eta_s^{k-1}\right) = 0 \quad (25)$$

For suitable collocation points, we use roots of shifting Legendre polynomial $L_{m-n+1}^*(\eta)$.
Also, by substituting formula (21) in the boundary conditions (9) we can obtain five equations as follows:

$$\sum_{i=0}^{m}(-1)^i c_i = 0 \qquad \sum_{i=0}^{m}(-1)^i d_i = 1 \qquad \sum_{i=0}^{m} d_i = 0 \qquad \sum_{i=0}^{m} L_i^{*'}(\eta_\infty)\, c_i = 0$$
$$1 - \sum_{i=0}^{m} L_i^{*''}(0)\, c_i + \lambda\sum_{i=0}^{m} L_i^{*'''}(0)\, c_i + \gamma\sum_{i=0}^{m} L_i^{*''''}(0)\, c_i = 0 \quad (26)$$

Eqs. (24–25), together with five equations of the boundary conditions (26), give a system of $(2m+2)$ algebraic equations which can be solved, for the unknowns c_i, d_i, $i = 0, 1, ..., m$ using the Newton iteration method.

5. Conclusion

In this paper, we used the Shifted Legendre collocation method. The boundary conditions at infinity pose a problem in some of heat transfer problems as well as many other nonlinear differential equations problems. Analytical solution of the problem cannot be obtained due to both the lack of linearity of the problem and the boundary conditions at infinity. In two-point boundary value problems where one point is infinity, two questions arise. The first one is that where infinity is and the second question is that when a satisfactory approximation to a solution has been obtained. In this study, a special case of free convection flow about a flat plate parallel to the direction of the generating body force problem, which is a nonlinear problem and possessing the boundary conditions at infinity, is solved semi-analytically for the first time without taking or estimating boundary conditions and it is easily seen that the solutions in the literature were made by using the boundary conditions for $f^{''}(0)$and $\theta^{'}(0)$given by Ostrach [2]. Even Ostrach obtained his solutions by estimating the values for $f^{''}(0)$and $\theta^{'}(0)$ [2]. In addition to these, it is clear that the obtained series is convergent. All numerical results are obtained using the Matlab program.

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Nomenclature

g – acceleration of gravity
T – temperature
T_∞ – free stream temperature
u – velocity component along x
v – velocity component along y
f – non-dimensional stream function
θ – non-dimensional temperature function
α – thermal diffusivity
β – thermal expansion coefficient of fluid
η – similarity variable
ν – kinematic viscosity