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Solving Fractional Integro-Differential Equations by Using Sumudu Transform Method and Hermite Spectral Collocation Method

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Abstract: In this paper we are looking forward to finding the approximate analytical solutions for fractional integro-differential equations by using Sumudu transform method and Hermite spectral collocation method. The fractional derivatives are described in the Caputo sense. The applications related to Sumudu transform method and Hermite spectral collocation method have been developed for differential equations to the extent of access to approximate analytical solutions of fractional integro-differential equations.

Keywords: Caputo derivative, integro-differential equations, hermite polynomials, sumudu transform.

1 Introduction

A lot of problems can be modeled by fractional integro-differential equations from various sciences and engineering applications. In addition to the fact that many problems cannot be found analytical solutions to them and therefore, once you get a solution is a result of a good result solutions, using numerical methods, will be very helpful. Recently, several numerical methods to solve fractional integro-differential equations (FIDEs) [Zedan, Tantawy, Sayed et al. (2017); Oyedepo, Taiwo, Abubakar et al. (2016); Wang and Zhu (2017)] have been given. Since the example collocation method for solving the nonlinear fractional Langevin equation [Bhrawy and Alghamdi (2012); Yang, Chen and Huang (2014)]. A Chebyshev polynomials method is introduced in Bhrawy et al. [Bhrawy and Alofi (2013)], Doha et al. [Doha, Bhrawy and Ezz-Eldien (2011)], Irandoust-pakchin et al. [Irandoust-pakchin, Kheiri and Abdi-mazraeh (2013)] for solving multiterm fractional orders differential equations and nonlinear Volterra and Fredholm Integro-differential equations of fractional order. The authors in Rathore et al. [Rathore, Kumar, Singh et al. (2012)] applied variational iteration method for solving fractional Integro-differential equations with the nonlocal boundary conditions and more methods in Wang et al. [Wang, Han and Xie (2012)], Lin et al. [Lin, Gu and Young (2010)].

In this paper Sumudu transform method [Wang, Han and Xie (2012); Lin, Gu and Young (2010); Singh and Kumar (2011); Ganji (2006); Hashim, Chowdhurly and Mawa (2008);

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He (1999); Liao (2005); Amer, Mahdy and Youssef (2017)] and Hermite spectral collocation method [Andrews (1985); Solouma and Khader (2016); Bagherpoorfard and Ghassabzade (2013)]; Bojdi, Ahmadi-Asl and Aminataei (2013); Brill (2002); Bialecki (1993); Dyksen and Lynch (2000); He (1999)] is applied to solving fractional integro-differential equations.

In this paper, we are concerned with the numerical solution of the following linear fractional integro-differential equation [Bhrawy and Alofi (2013); Doha, Bhrawy and Ezz-Eldien (2011); Irandoust-pakchin, Kheiri and Abdi-mazraeh (2013); Mohammed (2014)]:

$$D^{\alpha}U(x)= f(x)+\int_0^1 K(x,t)U(t)dt, \qquad 0\le x,t\le 1, \tag{1}$$

with initial conditions:

$$U^{(i)}(0)=\delta_i,\ n-1<\alpha\le n,\ ,n\in N \tag{2}$$

where $D^{\alpha}U(x)$ indicates the α th Caputo fractional derivative of $U(t)$; $f(x)$, $K(x,t)$ are given functions, x and t are real variables varying in the interval [0, 1], and $U(x)$ is the unknown function to be determined.

The paper is structured in six sections. In section 2, we begin with an introduction to some necessary definitions of fractional calculus theory. In section 3 we describe the homotopy perturbation sumudu transform method., In section 4 we describe the Hermite spectral collocation method. In section 5, we present two examples to show the efficiency of using HPSTM and Hermite spectral collocation method to solve FDE $_s$ and also to compare our results with those obtained by other existing methods. Finally, relevant conclusions are drawn in section 6.

2 Basic definitions of fractional calculus

In this section, we present the basic definitions and properties of the fractional calculus theory, which are used further in this paper

Definition 1: A real function $f(t),\ t>0,$ is said to be in the space $C_{\alpha},\ \alpha\in \mathbb{R},$ if there exists a real number $p>\alpha$ such that $f(t)=t^{p}f_1(t)$ where $f_1(t)\in C[0,\infty),$ and it is said to be in the space C_{α}^{m} if $f^{m}\in C_{\alpha},\ m\in \mathbb{N}.$

Definition 2: The Caputo fractional derivative operator D^{α} of order α is defined in the following form [El-Sayed and Salman (2013); El-Sayed and Salman (2013); Elsadany and Matouk (2015)]:

$$D^{\alpha}f(x)=\begin{cases}\dfrac{1}{\Gamma(m-\alpha)}\displaystyle\int_0^x \frac{f^{(m)}(\xi)}{(x-\xi)^{\alpha-m+1}}d\xi, & 0\le m-1<\alpha<m,\\ f^{(m)}(x), & \alpha=m\in N.\end{cases} \tag{3}$$

Similar to integer-order differentiation, The Caputo fractional derivative operator is linear

$$D^{\alpha}(c_1p(t)+c_2q(t))=c_1D^{\alpha}p(t)+c_2D^{\alpha}q(t),$$

where c_1 and c_2 are constants. For the Caputo's derivative we have $D^{\alpha}c = 0, c$ is a constant [Andrews (1985); Funaro (1992)].

$$D^{\alpha}x^{n} = \begin{cases} 0. & for n \in N_0 and n < \lceil \alpha \rceil; \\ \dfrac{\Gamma(n+1)}{\Gamma(n+1-\alpha)} x^{n-\alpha}, & for n \in N_0 and n \geq \lceil \alpha \rceil. \end{cases} \tag{4}$$

Definition 3: The Sumudu transform is defined over the set of functions [Singh and Kumar (2011); Ganji (2006)]

$$A = \left\{ f(t) \middle| \exists M, \tau_1, \tau_2 > 0, |f(t)| < Me^{\frac{|t|}{\tau_j}}, if t \in (-1)^j \times [0, \infty) \right\},$$

by the following formula:

$$\bar{f}(u) = S[f(t)] = \int_0^{\infty} f(ut)e^{-t}dt, \quad where \quad u \in (\tau_1, \tau_2) \tag{5}$$

where

Some special properties of the sumudu transform are as follows [Belgacem and Karaballi (2006)]:

1. $S[1] = 1;$
2. $S[t] = u \quad ;$
3. $S\left[\dfrac{t^m}{\Gamma(m+1)}\right] = u^m; \qquad m > 0$
4. $S\left[\dfrac{t^m}{(m)!}\right] = u^m; \qquad m = 1, 2, \cdots$
5. $If \ (f * g)(t) = \int_0^t f(\tau)g(t-\tau)d\tau \ then \ S(f * g)(t) = uF(u)G(u).$

Definition 4: The Sumudu transform of Caputo fractional derivative is defined as follows [Amer, Mahdy and Youssef (2017); Belgacem and Karaballi (2006)]:

$$S\left[D_t^{\alpha} f(t)\right] = u^{\alpha} S[f(t)] - \sum_{k=0}^{m-1} u^{-\alpha+k} f^{(k)}(0), \ m-1 < \alpha \leq m. \tag{6}$$

Theorem: [Singh and Kumar (2011); Amer, Mahdy and Youssef (2017)]

$$S\left[f^{(n)}(t)\right] = u^{-n}\left[F(u) - \sum_{k=0}^{n-1} u^k f^k(0)\right] \quad for\, n \geq 1 \tag{7}$$

At very special case for $n = 1$

$$S\left[F'(t)\right] = \frac{1}{u}\left[F(u) - F(0)\right].$$

This theorem is very important to calculate approximate solution of the problems and for more details in Singh et al. [Singh and Kumar (2011)], Amer et al. [Amer, Mahdy and Youssef (2017)]

Definition 5: The Hermite polynomials are given by Andrews [Andrews (1985)], Solouma et al. [Solouma and Khader (2016)], Bagherpoorfard et al. [Bagherpoorfard and Ghassabzade (2013)], Bojdi et al. [Bojdi, Ahmadi-Asl and Aminataei (2013)], Brill [Brill (2002)], Bialecki [Bialecki (1993)], Dyksen et al. [Dyksen and Lynch (2000)], He [He (1999)]:

$$H_n(y) = (-1)^n e^{y^2} \frac{d^n}{dz^n} e^{-y^2} \tag{8}$$

A lot of the properties of these polynomials are:

The Hermite polynomials evaluated at zero argument $H_n(0)$ and are have called Hermite number as follows: [Andrews (1985); Solouma and Khader (2016)]

$$H_n(0) = \begin{cases} 0, & if \quad n \, is \, odd \\ (-1)^{\frac{n}{2}} 2^{\frac{n}{2}} (n-1)!! & if \quad n \, is \, even \end{cases} \tag{9}$$

Where $(n-1)!!$ is the double factorial. The polynomials $H_n(y)$ are orthogonal with respect to the weight function $\omega(y) = e^{-y^2}$ with the following condition: [Andrews (1985)]

$$\int_{-\infty}^{\infty} H_n(y) H_m(y) \omega(y) dy = \sqrt{\pi}\, 2^n n! \delta_{nm}. \tag{10}$$

3 The homotopy perturbation sumudu transform method

In order to elucidate the solution procedure of this method, we consider a general fractional nonlinear differential equation of the form [Singh and Kumar (2011); Ganji (2006); Hashim, Chowdhurly and Mawa (2008); He (1999); Liao (2005); Amer, Mahdy and Youssef (2017)]:

$$D_*^{\alpha} y(t) + \mathbf{L} y(t) + \mathbf{N} y(t) = q(t), \tag{11}$$

with $m-1 < \alpha \le m,$ and subject to the initial condition

$$y^j(0) = c_j, \quad j = 0,1,\dots,m-1, \tag{12}$$

where $D_*^{\alpha} y(t)$ is the Caputo fractional derivative, $q(t)$ is the source term, $\mathbf{L}$ is the linear operator and $\mathbf{N}$ is the general nonlinear operator.

Applying the Sumudu transform (denoted throughout this paper by S) on both sides of Eq. (11), we have

$$S\left[D_*^{\alpha} y(t)\right] + S\left[\mathbf{L} y(t) + \mathbf{N} y(t)\right] = S\left[q(t)\right].$$

Using the property of the Sumudu transform and the initial conditions in Eq. (12), we have

$$S\left[y(t)\right] = \sum_{k=0}^{m-1} u^{-\alpha+k} y^k(0) + u^{\alpha} S\left[q(t)\right] - u^{\alpha} S\left[\mathbf{L} y(t) + \mathbf{N} y(t)\right], \tag{13}$$

Operating with the Sumudu inverse on both sides of Eq. (13) we get

$$y(t) = G(t) - S^{-1}\left[u^{\alpha} S\left[\mathbf{L}y(t) + \mathbf{N}y(t)\right]\right]. \tag{14}$$

Where $G(t)$ represents the term arising from the source term and the prescribed initial conditions. Now, playing the classical perturbation technique. And assuming that the solution of Eq. (14) is in the form

$$y(t) = \sum_{m=0}^{\infty} p^m y_m(t), \tag{15}$$

where $p \in [0,1]$ is the homotopy parameter. The nonlinear term of Eq. (14) can be decomposed as

$$\mathbf{N}y(t) = \sum_{m=0}^{\infty} p^m A_m(t), \tag{16}$$

for some Adomian's polynomials A_m, which can be calculated with the formula [Ghorbani (2009); Jafari and Daftardar-Gejji (2006)]

$$A_m = \frac{1}{m!}\frac{d^m}{dp^m}\left[\mathbf{N}\left(\sum_{i=0}^{\infty} p^i y_i(t)\right)\right]_{p=0},\ n = 0,1,2,\ \ldots \tag{17}$$

Substituting Eq. (15) and (17) in Eq. (14), we get

$$\sum_{m=0}^{\infty} p^m y_m(t) = G(t) - pS^{-1}\left[u^{\alpha} S\left[\mathbf{L}\left(\sum_{m=0}^{\infty} p^m y_m(t)\right) + \sum_{m=0}^{\infty} p^m A_m\right]\right]. \tag{18}$$

Equating the terms with identical powers of p, we can obtain a series of equations as the follows:

$$\begin{aligned} p^0 &: y_0(t) = G(t), \\ p^1 &: y_1(t) = -S^{-1}\left[u^{\alpha} S\left[\mathbf{L}y_0(t) + A_0\right]\right], \\ p^2 &: y_2(t) = -S^{-1}\left[u^{\alpha} S\left[\mathbf{L}y_1(t) + A_1\right]\right], \\ p^3 &: y_3(t) = -S^{-1}\left[u^{\alpha} S\left[\mathbf{L}y_2(t) + A_2\right]\right], \end{aligned} \tag{19}$$

$$\vdots$$

Finally, we approximate the analytical solution $y(t)$ by truncated series as

$$y(t) = \lim_{M\to\infty} \sum_{m=0}^{M} p^m y_m(t) \tag{20}$$

4 Basic idea of hermite collocation method

In this section the Hermite collocation method is applied to study the numerical solution of the fractional Integro-differential (1).

This method is based on approximating the unknown function $u(x)$ as

$$u_n(x) = \sum_{n=0}^{m} a_n H_n(x) \tag{21}$$

Where $H_n(x)$ is the Hermite polynomials and a_n are constant

At first by Substituting (21) into (1) we obtain

$$D^{\alpha}\left(\sum_{n=0}^{m} a_n H_n(x)\right) = f(x) + \int_0^1 K(x,t)\left[\sum_{n=0}^{m} a_n H_n(x)\right]dt \tag{22}$$

Hence the residual equation is defined as:

$$R(x,a_0,a_1,\cdots,a_n) = D^{\alpha}\left(\sum_{n=0}^{m} a_n H_n(x)\right) - f(x) - \int_0^1 K(x,t)\left[\sum_{n=0}^{m} a_n H_n(x)\right]dt \tag{23}$$

Second let

$$s(x,a_0,a_1,\cdots,a_n) = \int_0^1 [R(x,a_0,a_1,\cdots,a_n)]^2\,\omega(x)dx \tag{24}$$

where $\omega(x)$ is the positive weight function defined on the interval [0, 1]. In this work we take $\omega(x)$ = 1 for simplicity.Thus

$$s(x,a_0,a_1,\cdots,a_n) = \int_0^1 \left\{D^{\alpha}\left(\sum_{n=0}^{m} a_n H_n(x)\right) - f(x) - \int_0^1 K(x,t)\left[\sum_{n=0}^{m} a_n H_n(x)\right]dt\right\}^2 dx \tag{25}$$

So, finding the values of $a_n, n = 0,1,\cdots,m$, which minimize S is equivalent to finding the best approximation for the solution of the fractional Integro-differential Eq. (1).

The minimum value of S is obtained by setting

$$\frac{\partial s}{\partial a_n} = 0, \;\; n = 0,1,\cdots,m \tag{26}$$

By applying (26) in (25) we have :

$$\int_0^1 \left\{D^{\alpha}\left(\sum_{n=0}^{m} a_n H_n(x)\right) - f(x) - \int_0^1 K(x,t)\left[\sum_{n=0}^{m} a_n H_n(x)\right]dt\right\} \times \left\{D^{\alpha} H_n(x) - \int_0^1 K(x,t) H_n(x)dt\right\}dx \tag{27}$$

By evaluating the above equation for $n = 0,1,\cdots,m$ we can obtain a system of (n+1) linear equations with (n+1) unknown coefficients a_n, after calculate the coefficient a_n we substitute in Eq. (21) then we get the solution of $U(x)$.

5 Applications

In this section, to illustrate the method and to show the ability of the method two examples are presented.

Example (1): Cosider the fractional integro-differential equations as

$$D^{\frac{1}{3}}y(x)=\frac{9}{5\Gamma\left(\frac{2}{3}\right)}x^{\frac{5}{3}}-\frac{7}{40}x+\frac{1}{4}\int_0^1 x(1-t)(y(t))^2\,dt \tag{28}$$

subject to

$$y(0)=1 \tag{29}$$

(i) **First by using Sumudu transform method**

By taking the Sumudu transform on both sides of Eq. (28), thus we get

$$S\left[D^{\frac{1}{3}}y(x)\right]=S\left[\frac{9}{5\Gamma\left(\frac{2}{3}\right)}x^{\frac{5}{3}}-\frac{7}{40}x+\frac{1}{4}\int_0^1 x(1-t)(y(t))^2\,dt\right]$$

$$S[y(x)]=y(0)+u^{\frac{1}{3}}.S\left[\frac{9}{5\Gamma\left(\frac{2}{3}\right)}x^{\frac{5}{3}}-\frac{7}{40}x+\frac{1}{4}\int_0^1 x(1-t)(y(t))^2\,dt\right] \tag{30}$$

Using the property of the Sumudu transform and the initial condition in Eq. (30), we have

$$S[y(x)]=1+2u^2-\frac{7}{40}u^{\frac{4}{3}}+\frac{1}{4}u^{\frac{7}{3}}S\left[y^2(t)\right] \tag{31}$$

Operating with the Sumudu inverse on both sides of Eq. (31) we get

$$y(x)=1+x^2-\frac{7}{40\Gamma\left(\frac{7}{3}\right)}x^{\frac{4}{3}}+\frac{1}{4}S^{-1}\left[u^{\frac{7}{3}}S\left[y^2(t)\right]\right] \tag{32}$$

By assuming that

$$y(x)={}_{n=0}^{\infty}\ y_n(x) \tag{33}$$

By substituting Eq. (33) in Eq. (32) we have

$${}_{n=0}^{\infty}\, y_n(x)=1+x^2-\frac{7}{40\Gamma\left(\frac{7}{3}\right)}x^{\frac{4}{3}}+\frac{1}{4}S^{-1}\left[u^{\frac{7}{3}}S[A_n]\right] \tag{34}$$

Where A_n, B_n are Adomian polynomials that represent nonlinear term. So Adomian polynomials are given as follows:

$$A_n(x)=y^2(t),$$

The few components of the Adomian polynomials are given as follows:

$$A_0 = y_0^2$$
$$A_1 = 2y_0y_1$$
$$A_2 = 2y_0y_2 + y_1^2$$
$$\vdots$$

Then we have

$$y_0 = 1 + x^2 - \frac{7}{40\Gamma\left(\frac{7}{3}\right)}x^{\frac{4}{3}}$$

$$A_0 = 1 + 2x^2 + x^4 - \frac{7}{20\Gamma\left(\frac{7}{3}\right)}x^{\frac{4}{3}} - \frac{7}{40\Gamma\left(\frac{7}{3}\right)}x^{\frac{10}{3}} + \frac{49}{1600\left(\Gamma\left(\frac{7}{3}\right)\right)^2}x^{\frac{8}{3}}$$

$$y_{k+1}(x) = \frac{1}{4}S^{-1}\left[u^{\frac{7}{3}}S[A_k]\right] \tag{35}$$

$$y_1(x) = \frac{1}{4}S^{-1}\left[u^{\frac{7}{3}}S[A_0]\right]$$

$$y_1 = \frac{1}{\Gamma\left(\frac{10}{3}\right)}x^{\frac{7}{3}} + \frac{4}{\Gamma\left(\frac{16}{3}\right)}x^{\frac{13}{3}} + \frac{24}{\Gamma\left(\frac{22}{3}\right)}x^{\frac{19}{3}} - \frac{7}{20\Gamma\left(\frac{14}{3}\right)}x^{\frac{11}{3}} - \frac{49}{18\Gamma\left(\frac{20}{3}\right)}x^{\frac{17}{3}} + \frac{49\Gamma\left(\frac{11}{3}\right)}{192000\left(\Gamma\left(\frac{7}{3}\right)\right)^2}x^5$$

$$\vdots$$

Since

$$y(x) = y_1 + y_2 + y_3 + \cdots$$

then

$$y(x) = 1 - \frac{7}{40\Gamma\left(\frac{7}{3}\right)}x^{\frac{4}{3}} + x^2 + \frac{1}{\Gamma\left(\frac{10}{3}\right)}x^{\frac{7}{3}} - \frac{7}{20\Gamma\left(\frac{14}{3}\right)}x^{\frac{11}{3}} + \frac{4}{\Gamma\left(\frac{16}{3}\right)}x^{\frac{13}{3}} - \frac{49}{18\Gamma\left(\frac{20}{3}\right)}x^{\frac{17}{3}} + \frac{24}{\Gamma\left(\frac{22}{3}\right)}x^{\frac{19}{3}}$$
$$+ \frac{49\Gamma\left(\frac{11}{3}\right)}{192000\left(\Gamma\left(\frac{7}{3}\right)\right)^2}x^5 + \ldots$$

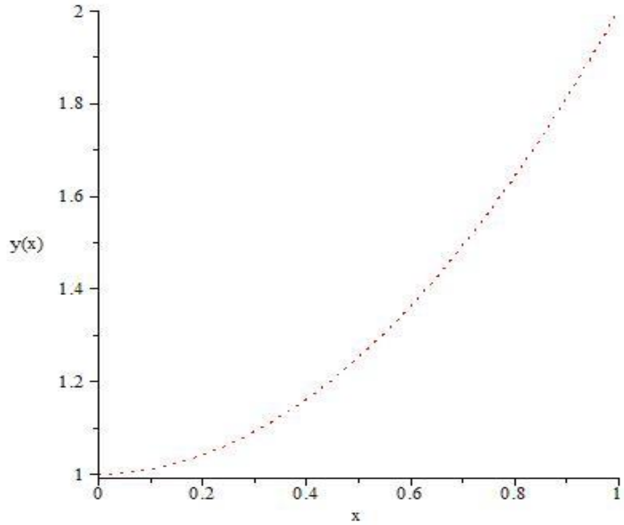

Figure 1: The behavior of y(x) by HPSTM

(ii) **By sing Hermite spectral collocation method**

First By assuming the approximate of the solution of $y(x)$ with m=2 as:

$$y(x)=\sum_{n=0}^{2}a_nH_n(x),\quad y(t)=\sum_{n=0}^{2}a_nH_n(t) \tag{36}$$

Where $H_n(x)$ is the Hermite polynomials and a_n are constant

Second by Substituting (36) into (28) we obtain

$$D^{\frac{1}{3}}\left(\sum_{n=0}^{2}a_nH_n(x)\right)=\frac{9}{5\Gamma\left(\frac{2}{3}\right)}x^{\frac{5}{3}}-\frac{7}{40}x+\frac{1}{4}\int_0^1 x(1-t)\left(\sum_{n=0}^{2}a_nH_n(t)\right)^2 dt \tag{37}$$

Hence the residual equation is defined as:

$$R(x,a_0,a_1,\cdots,a_n)=D^{\frac{1}{3}}\left(\sum_{n=0}^{2}a_nH_n(x)\right)-\frac{9}{5\Gamma\left(\frac{2}{3}\right)}x^{\frac{5}{3}}+\frac{7}{40}x-\frac{1}{4}\int_0^1 x(1-t)\left(\sum_{n=0}^{m}a_nH_n(t)\right)^2 dt \tag{38}$$

By substitutinn $H_n(x),\ H_n(t)$ and Eq. (4) in Eq. (38) we get

$$R(x,a_0,a_1,\cdots,a_n)=\frac{8a_2}{\Gamma\left(\frac{8}{9}\right)}x^{\frac{5}{6}}+\frac{2a_1}{\Gamma\left(\frac{5}{6}\right)}x^{\frac{2}{3}}-\frac{9}{5.\Gamma\left(\frac{2}{3}\right)}x^{\frac{5}{3}}-\frac{x}{4}\left[\frac{2}{3}a_1+\frac{16}{15}a_2+\frac{1}{3}a_1^2+\frac{8}{15}a_2^2+\frac{4}{5}a_1a_2-\frac{1}{5}\right] \tag{39}$$

Second let

$$s(x,a_0,a_1,\cdots,a_n)=\int_0^1[R(x,a_0,a_1,\cdots,a_n)]^2\omega(x)dx \tag{40}$$

where $\omega(x)$ is the positive weight function defined on the interval [0, 1]. In this work we take $\omega(x)$=1 for simplicity.Thus

$$s(x,a_0,a_1,\cdots,a_n)=\int_0^1\left\{\frac{8a_2}{\Gamma\left(\frac{8}{9}\right)}x^{\frac{5}{6}}+\frac{2a_1}{\Gamma\left(\frac{5}{6}\right)}x^{\frac{2}{3}}-\frac{9}{5.\Gamma\left(\frac{2}{3}\right)}x^{\frac{5}{3}}-\frac{x}{4}\left[\frac{2}{3}a_1+\frac{16}{15}a_2+\frac{1}{3}a_1^2+\frac{8}{15}a_2^2+\frac{4}{5}a_1a_2-\frac{1}{5}\right]\right\}^2dx \tag{41}$$

The minimum value of S is obtained by setting

$$\frac{\partial s}{\partial a_n}=0,\ \ n=0,1,2 \tag{42}$$

By applying (42) in (41) we have:

$$25.6a_2+0.779a_1-0.2a_1^2-0.09a_2^2-0.3a_1a_2-0.4$$
$$+\frac{1}{48}\left(\frac{2}{3}a_1+\frac{16}{15}a_2+\frac{1}{3}a_1^2+\frac{8}{15}a_2^2+\frac{4}{5}a_1a_2-\frac{1}{5}\right)\left(\frac{2}{3}+\frac{2}{3}a_1+\frac{4}{5}a_2\right)=0 \tag{43}$$

$$19.3a_2+4a_1-0.1a_1^2-0.37a_2^2-0.74a_1a_2-3.2$$
$$+\frac{1}{48}\left(\frac{2}{3}a_1+\frac{16}{15}a_2+\frac{1}{3}a_1^2+\frac{8}{15}a_2^2+\frac{4}{5}a_1a_2-\frac{1}{5}\right)\left(\frac{16}{15}+\frac{16}{15}a_1+\frac{4}{5}a_2\right)=0 \tag{44}$$

From the initial condhtion y(0)=1 and from Eq. (7) we get

$$a_0-2a_2=1 \tag{45}$$

By solving the Eq. (43)-(45) we get the values of a_0,a_1,a_2 and substituting in Eq. (36) we get the solution as series:

$$y(x)=1+x^2 \tag{46}$$

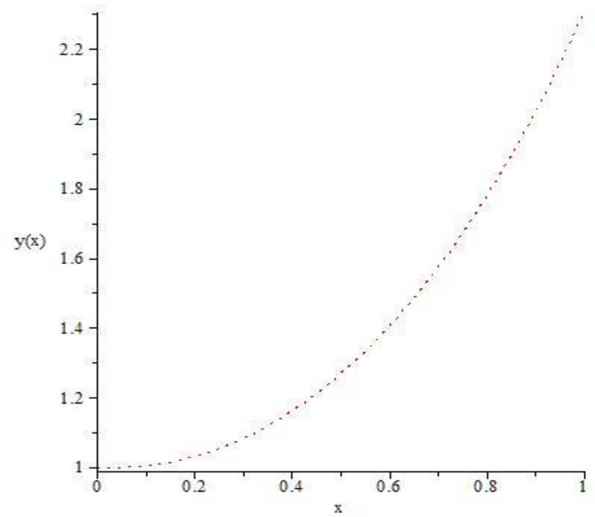

Figure 2: The behavior of y(x) by Hermite collocation method

It is no doubt that the efficiency of this approach is greatly enhanced by the calculation further terms of $y(x)$ by using by using Sumudu transform method and Hermite spectral collocation method.As shown in Fig. 1 and Fig. 2.

Example (2): Consider the systems of fractional integro-differential type as :

$$D^{\frac{2}{3}}u(x)=\frac{3\sqrt{3}\Gamma\left(\frac{2}{3}\right)}{2\pi}x^{\frac{1}{3}}-\frac{1}{6}x+\int_0^1 2xt(u(t)+v(t))dt \tag{47}$$

$$D^{\frac{2}{3}}v(x)=\frac{9\sqrt{3}\Gamma\left(\frac{2}{3}\right)}{4\pi}x^{\frac{4}{3}}+\frac{5}{6}x^3+\int_0^1 x^3(u(t)-v(t))dt \tag{48}$$

subject to

$$u(0)=-1\ \ ,\ v(0)=0, \tag{49}$$

By using the properities of Gamma function of the two Eq. (47), (48) become

$$D^{\frac{2}{3}}u(x)=\frac{3}{\Gamma\left(\frac{1}{3}\right)}x^{\frac{1}{3}}-\frac{1}{6}x+\int_0^1 2xt(u(t)+v(t))dt50$$

$$D^{\frac{2}{3}}v(x)=\frac{9}{2\Gamma\left(\frac{1}{3}\right)}x^{\frac{4}{3}}+\frac{5}{6}x^3+\int_0^1 x^3(u(t)-v(t))dt \tag{50}$$

(i) **First by using Sumudu transform method**

By taking the Sumudu transform on both sides of Eq. (50), thus we get

$$\begin{cases} S\left[D^{\frac{2}{3}}u(x)\right]=S\left[\dfrac{3}{\Gamma\left(\frac{1}{3}\right)}x^{\frac{1}{3}}-\dfrac{1}{6}x+\int_0^1 2xt(u(t)+v(t))dt\right], \\ S\left[D^{\frac{2}{3}}v(x)\right]=S\left[\dfrac{9}{2\Gamma\left(\frac{1}{3}\right)}x^{\frac{4}{3}}+\dfrac{5}{6}x^3+\int_0^1 x^3(u(t)-v(t))dt\right]. \end{cases}$$

$$\begin{aligned} &S\left[D^{\frac{2}{3}}u(x)\right]=u^{\frac{1}{3}}-\frac{1}{6}u+S\left[\int_0^1 2xt(u(t)+v(t))dt\right], \\ &S\left[D^{\frac{2}{3}}v(x)\right]=2u^{\frac{4}{3}}+5u^3+S\left[\int_0^1 x^3(u(t)-v(t))dt\right]. \end{aligned} \tag{51}$$

Using the property of the Sumudu transform and the initial condition in Eq. (49), we have

$$\begin{cases} S[u]=u(0)+u-\dfrac{1}{6}u^{\frac{5}{3}}+u^{\frac{2}{3}}S\left[\int_0^1 2xt(u(t)+v(t))dt\right], \\ S[v]=v(0)+2u^2+5u^{\frac{11}{3}}+u^{\frac{2}{3}}S\left[\int_0^1 x^3(u(t)-v(t))dt\right] \end{cases}$$

and

$$\begin{cases} S[u]=-1+u-\dfrac{1}{6}u^{\frac{5}{3}}+2u^{\frac{8}{3}}S[u(x)+v(x)], \\ S[v]=2u^2+5u^{\frac{11}{3}}+6u^{\frac{14}{3}}S[u(x)-v(x)] \end{cases} \tag{52}$$

Operating with the Sumudu inverse on both sides of Eq. (52) we get

$$\begin{cases} u(x) = -1 + x - \dfrac{1}{6\Gamma\left(\frac{8}{3}\right)} x^{\frac{5}{3}} + 2S^{-1}\left[u^{\frac{8}{3}} S[u(x) + v(x)] \right], \\ v(x) = x^2 + \dfrac{5}{\Gamma\left(\frac{14}{3}\right)} x^{\frac{11}{3}} + 6S^{-1}\left[u^{\frac{14}{3}} S[u(x) - v(x)] \right]. \end{cases} \quad (53)$$

By assuming that

$$u(x) = {}_{n=0}^{\infty} u_n(x) \quad , v(x) = {}_{n=0}^{\infty} v_n(x) \quad (54)$$

By substituting Eq. (54) in Eq. (53) we have

$$\begin{cases} {}_{n=0}^{\infty} u_n(x) = -1 + x - \dfrac{1}{6\Gamma\left(\frac{8}{3}\right)} x^{\frac{5}{3}} + 2S^{-1}\left[u^{\frac{8}{3}} S\left[{}_{n=0}^{\infty} u_n(x) + {}_{n=0}^{\infty} v_n(x)\right] \right], \\ {}_{n=0}^{\infty} v_n(x) = x^2 + \dfrac{5}{\Gamma\left(\frac{14}{3}\right)} x^{\frac{11}{3}} + 6S^{-1}\left[u^{\frac{14}{3}} S\left[{}_{n=0}^{\infty} u_n(x) + {}_{n=0}^{\infty} v_n(x)\right] \right]. \end{cases} \quad (55)$$

Then we have

$$u_0(x) = -1 + x - \frac{1}{6\Gamma\left(\frac{8}{3}\right)} x^{\frac{5}{3}}$$

$$v_0(x) = x^2 + \frac{5}{\Gamma\left(\frac{14}{3}\right)} x^{\frac{11}{3}}$$

$$U_{k+1}(x) = 2S^{-1}\left[u^{\frac{8}{3}} S\left[u_k(x) + v_k(x) \right] \right] \quad (56)$$

$$V_{k+1}(x) = 6S^{-1}\left[u^{\frac{14}{3}} S[u_k(x) - v_k(x)] \right]$$

Then

$$u_1 = 2S^{-1}\left[u^{\frac{8}{3}} S[u_0(x) + v_0(x)] \right]$$

$$v_1 = 6S^{-1}\left[u^{\frac{14}{3}} S[u_0(x) - v_0(x)]\right]$$

$$u_1 = \frac{-2}{\Gamma\left(\frac{11}{3}\right)} x^{\frac{8}{3}} + \frac{2}{\Gamma\left(\frac{14}{3}\right)} x^{\frac{11}{3}} + \frac{1}{3\Gamma\left(\frac{16}{3}\right)} x^{\frac{13}{3}} + \frac{4}{\Gamma\left(\frac{17}{3}\right)} x^{\frac{14}{3}} + \frac{10}{\Gamma\left(\frac{22}{3}\right)} x^{\frac{19}{3}}$$

$$v_1 = \frac{-6}{\Gamma\left(\frac{17}{3}\right)} x^{\frac{14}{3}} + \frac{6}{\Gamma\left(\frac{20}{3}\right)} x^{\frac{17}{3}} + \frac{1}{\Gamma\left(\frac{22}{3}\right)} x^{\frac{19}{3}} - \frac{12}{\Gamma\left(\frac{23}{3}\right)} x^{\frac{20}{3}} - \frac{30}{\Gamma\left(\frac{28}{3}\right)} x^{\frac{25}{3}}$$

$$\vdots$$

Since

$$u(x) = u_0 + u_1 + u_2 + \cdots$$

$$v(x) = v_1 + v_2 + v_3 + \cdots$$

then

$$u(x) = -1 + x - \frac{1}{6\Gamma\left(\frac{8}{3}\right)} x^{\frac{5}{3}} - \frac{2}{\Gamma\left(\frac{11}{3}\right)} x^{\frac{8}{3}} + \frac{2}{\Gamma\left(\frac{14}{3}\right)} x^{\frac{11}{3}} + \frac{1}{3\Gamma\left(\frac{16}{3}\right)} x^{\frac{13}{3}} + \frac{4}{\Gamma\left(\frac{17}{3}\right)} x^{\frac{14}{3}} + \frac{10}{\Gamma\left(\frac{22}{3}\right)} x^{\frac{19}{3}} + \cdots$$

$$v(x) = x^2 + \frac{5}{\Gamma\left(\frac{14}{3}\right)} x^{\frac{11}{3}} - \frac{6}{\Gamma\left(\frac{17}{3}\right)} x^{\frac{14}{3}} + \frac{6}{\Gamma\left(\frac{20}{3}\right)} x^{\frac{17}{3}} + \frac{1}{\Gamma\left(\frac{22}{3}\right)} x^{\frac{19}{3}} - \frac{12}{\Gamma\left(\frac{23}{3}\right)} x^{\frac{20}{3}} - \frac{30}{\Gamma\left(\frac{28}{3}\right)} x^{\frac{25}{3}} + \cdots$$

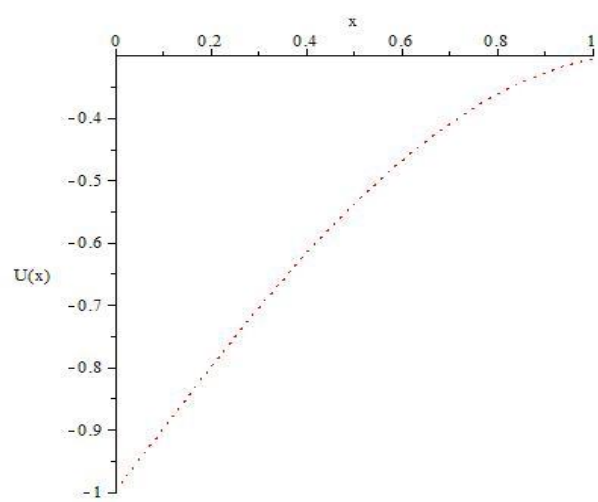

Figure 3: The behavior of u(x) by HPSTM

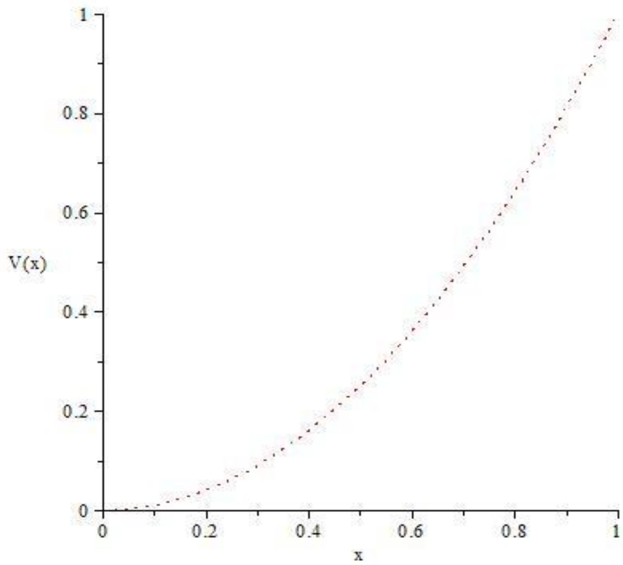

Figure 4: The behavior of v(x) by HPSTM

(ii) **By sing Hermite spectral collocation method**

First By assuming the approximate of the solution of $y(x)$ with m=2 as:

$$u(x)=\sum_{n=0}^{2}c_nH_n(x),\ u(t)=\sum_{n=0}^{2}c_nH_n(t)57 \tag{57}$$

$$v(x)=\sum_{n=0}^{2}a_nH_n(x),\ v(t)=\sum_{n=0}^{2}a_nH_n(t)$$

Where $H_n(x)$ is the Hermite polynomials and a_n are constant

Second by Substituting (57) into (50) we obtain

$$D^{\frac{2}{3}}\sum_{n=0}^{2}c_nH_n(x)=\frac{3}{\Gamma\left(\frac{1}{3}\right)}x^{\frac{1}{3}}-\frac{1}{6}x+\int_0^1 2xt\left(\sum_{n=0}^{2}c_nH_n(t)+\sum_{n=0}^{2}a_nH_n(t)\right)dt58 \tag{58}$$

$$D^{\frac{2}{3}}\sum_{n=0}^{2}a_nH_n(x)=\frac{9}{2\Gamma\left(\frac{1}{3}\right)}x^{\frac{4}{3}}+\frac{5}{6}x^3+\int_0^1 x^3\left(\sum_{n=0}^{2}c_nH_n(t)-\sum_{n=0}^{2}a_nH_n(t)\right)dt$$

Hence the residual equation is defined as:

$$R\left(x,c_0,c_1,\cdots,c_n\right)=D^{\frac{2}{3}}\sum_{n=0}^{2}c_nH_n\left(x\right)-\frac{3}{\Gamma\left(\frac{1}{3}\right)}x^{\frac{1}{3}}+\frac{1}{6}x-\int_0^1 2xt\left(\sum_{n=0}^{2}c_nH_n\left(t\right)+\sum_{n=0}^{2}a_nH_n\left(t\right)\right)dt \quad (59)$$

$$R\left(x,a_0,a_1,\cdots,a_n\right)=D^{\frac{2}{3}}\sum_{n=0}^{2}a_nH_n\left(x\right)-\frac{9}{2\Gamma\left(\frac{1}{3}\right)}x^{\frac{4}{3}}-\frac{5}{6}x^3-\int_0^1 x^3\left(\sum_{n=0}^{2}c_nH_n\left(t\right)-\sum_{n=0}^{2}a_nH_n\left(t\right)\right)dt$$

By substitutinn $H_n(x)$, $H_n(t)$ and Eq. (4) in Eq. (59) we get

$$R\left(x,c_0,c_1,\cdots,c_n\right)=\frac{18c_2}{\Gamma\left(\frac{1}{3}\right)}x^{\frac{4}{3}}+\frac{6c_1}{\Gamma\left(\frac{1}{3}\right)}x^{\frac{1}{3}}+\frac{7}{6}x-\frac{3}{\Gamma\left(\frac{1}{3}\right)}x^{\frac{1}{3}}-2x\left(a_2+c_2\right)-\frac{4}{3}x\left(a_1+c_1\right) \quad (60)$$

$$R\left(x,a_0,a_1,\cdots,a_n\right)=\frac{18a_2}{\Gamma\left(\frac{1}{3}\right)}x^{\frac{4}{3}}+\frac{6a_1}{\Gamma\left(\frac{1}{3}\right)}x^{\frac{1}{3}}+\frac{1}{6}x^3-\frac{9}{2\Gamma\left(\frac{1}{3}\right)}x^{\frac{4}{3}}-\frac{4}{3}x^3\left(c_2-a_2\right)-\frac{4}{3}x^3\left(c_1-a_1\right)$$

Second let

$$S\left(x,c_0,c_1,\cdots,c_n\right)=\int_0^1\left[R\left(x,c_0,c_1,\cdots,c_n\right)\right]^2\omega\left(x\right)dx \quad (61)$$

$$S\left(x,a_0,a_1,\cdots,a_n\right)=\int_0^1\left[R\left(x,a_0,a_1,\cdots,a_n\right)\right]^2\omega\left(x\right)dx$$

where $\omega(x)$ is the positive weight function defined on the interval [0, 1]. In this work we take $\omega(x)$=1 for simplicity.Thus

$$S(x,c_0,c_1,\cdots,c_n)=\int_0^1\left\{\frac{18c_2}{\Gamma\left(\frac{1}{3}\right)}x^{\frac{4}{3}}+\frac{6c_1}{\Gamma\left(\frac{1}{3}\right)}x^{\frac{1}{3}}+\frac{7}{6}x-\frac{3}{\Gamma\left(\frac{1}{3}\right)}x^{\frac{1}{3}}-2x(a_2+c_2)-\frac{4}{3}x(a_1+c_1)\right\}^2dx, \quad (62)$$

$$S\left(x,a_0,a_1,\cdots,a_n\right)=\int_0^1\left\{\frac{18a_2}{\Gamma\left(\frac{1}{3}\right)}x^{\frac{4}{3}}+\frac{6a_1}{\Gamma\left(\frac{1}{3}\right)}x^{\frac{1}{3}}+\frac{1}{6}x^3-\frac{9}{2\Gamma\left(\frac{1}{3}\right)}x^{\frac{4}{3}}-\frac{4}{3}x^3\left(c_2-a_2\right)-\frac{4}{3}x^3\left(c_1-a_1\right)\right\}^2dx$$

The minimum value of S is obtained by setting

$$\frac{\partial s}{\partial a_n}=0,\ \frac{\partial s}{\partial c_n}=0, n=0,1,2 \quad (63)$$

From the initial condhtion $u(0)=-1, v(0)=0$ and from Eq. (7) we get

$$c_0-2c_2=-1,\ a_0-2a_2=0 \quad (64)$$

By solving the equations produced from (63) with (64) we get the solution as series

$$u(x)=x-1,\ v(x)=x^2$$

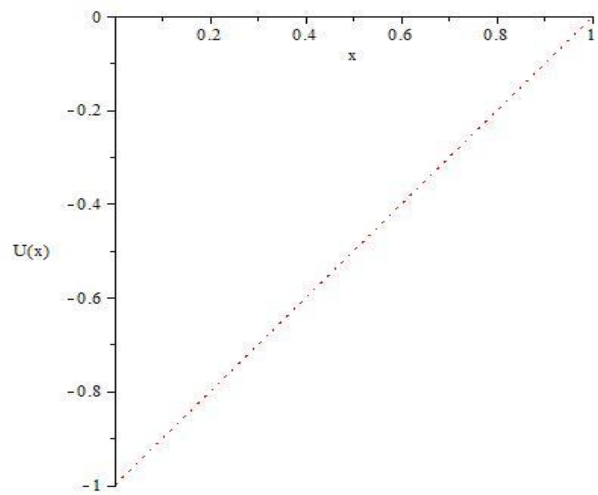

Figure 5: The behavior of u(x) by Hermite collocation method

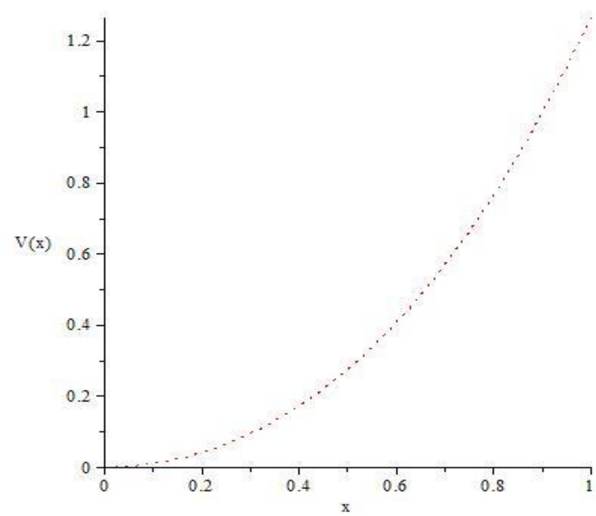

Figure 6: The behavior of v(x) by Hermite collocation method

It is no doubt that the efficiency of this approach is greatly enhanced by the calculation further terms of u$(x), v(x)$ by using by using Sumudu transform method and Hermite spectral collocation method. As In Fig. 3 and Fig. 4 show the The behavior of u$(x), v(x)$ by using Sumudu transform method and in Fig. 5 and Fig. 6. show the The behavior of u $(x), v(x)$ by using the Hermite collocation method.

6 Conclusions

The main aim of this paper is to know that the sumud transform method and Hermite spectral collocation method are of the most important and simplest methods used in solving linear and nonlinear differential equations. This method have been successfully applied to systems of fractional integro-differential equations.in this method we do not need to do the difficult computation for finding the Adomian polynomials. Generally speaking, the proposed method is promising and applicable to a broad class of linear and nonlinear problems in the theory of fractional calculus.

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