Geometrical explanation of the fractional complex transform and derivative chain rule for fractional calculus

Ji-Huan He a,*, S.K. Elagan b,1, Z.B. Li c

a National Engineering Laboratory for Modern Silk, College of Textile and Engineering, Soochow University, 199 Ren-ai Road, Suzhou 215123, China
b Mathematics & Statistics Department, Faculty of Science, Taif University, PO. 888, Saudi Arabia
c College of Mathematics and Information Science, Qujing Normal University, Qujing, Yunnan 655011, China

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A B S T R A C T
The fractional complex transform is suggested to convert a fractional differential equation with Jumarie’s modification of Riemann–Liouville derivative into its classical differential partner. Understanding the fractional complex transform and the chain rule for fractional calculus are elucidated geometrically.
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1. Introduction
Recently some effective methods for fractional calculus were appeared in open literature, such as the fractional complex transform [1–4], the homotopy perturbation method [5,6], the variational iteration method [7–12], the exp-function method [13], and the heat-balance integral method [14–16] and other analytical methods [17–22], among which the fractional complex transform [1–4] is the simplest approach, it is to convert the fractional differential equations into ordinary differential equations, making the solution procedure extremely simple.

Similar to wave transformation

\[ \xi = qt + px + ky + lz \]  

where \( p, q, k \) and \( l \) are constants, for nonlinear wave equations, e.g., the KdV equation, the fractional complex transform also admits a complex variable \( \xi \), instead of Eq. (1), defined as [1,2]

\[ \xi = \frac{qt^\alpha}{\Gamma(1+\alpha)} + \frac{px^\beta}{\Gamma(1+\beta)} + \frac{ky^\gamma}{\Gamma(1+\gamma)} + \frac{lz^\lambda}{\Gamma(1+\lambda)} \]  

where \( \alpha, \beta, \gamma, \) and \( \lambda \) are fractional orders.

Such transformation is valid only for general “wave” solutions for fractional differential equations. However, not every fractional differential equation has a “wave” solution, hence its application is limited.

In this Letter we suggest a modification to convert a fractional differential equation into its classical differential partner.

2. A modification of the fractional complex transform
Consider the following general fractional differential equation

\[ f(u, u_x^{(\alpha)}, u_y^{(\beta)}, u_z^{(\gamma)}) = 0, \quad 0 < \alpha \leq 1, \quad 0 < \beta \leq 1, \quad 0 < \gamma \leq 1, \quad 0 < \lambda \leq 1 \]  

where \( u_t^{(\alpha)} = D_t^\alpha u = D_t^\alpha u/DT^\alpha \) denotes Jumarie’s fractional derivation, which is a modified Riemann–Liouville derivative [23–26] defined as

\[ D_t^\alpha u(t, x, y, z) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_0^t (t-\xi)^{-\alpha} \]

\[ \times (u(\xi, x, y, z) - u(0, x, y, z)) d\xi \]  

where \( u \) is a continuous (but not necessarily differentiable) function.

The modified fractional complex transform reads

\[ T = \frac{qt^\alpha}{\Gamma(1+\alpha)} \]  

\[ X = \frac{px^\beta}{\Gamma(1+\beta)} \]
Consider a plane with fractal structure (see Fig. 1). The shortest path fractional order is equivalent to its fractional dimensions. Now, potential equations can best describe discontinuous media, and the correction yields Cantor-like sets, and its length can be expressed as

\[ \text{This idea leads to the fractional complex transform, Eqs. (5)–(8).} \]

\[ \frac{\partial^u u}{\partial t^\alpha} = \frac{\partial u}{\partial s} \frac{\partial^s s}{\partial t^\alpha}. \]

This chain rule is invalid.

Consider a counter example

\[ u(t) = t^2, \quad s(x) = x^\beta, \quad \beta > 0. \]

Then we have

\[ D_x^\alpha u(s(x)) = D_x^\alpha u(x^\beta) = D_x^\alpha x^\beta = \frac{x^{2\beta - \alpha}}{\Gamma(2\beta - \alpha + 1)} \Gamma(2\beta + 1). \]

We, therefore, find that

\[ \frac{\partial^u u}{\partial t^\alpha} \neq \frac{\partial u}{\partial s} \frac{\partial^s s}{\partial t^\alpha}. \]

This discrepancy arises in non-commutative property in fractional calculus, that is \( D_x^\alpha + D_y^\beta \neq D_x^\alpha D_y^\beta \).

The chain rule hereby is actually a fractal space change, e.g., the fractal curve “AB” in Fig. 1 is projected to Cantor-like sets in horizontal direction. From Fig. 1, we have

\[ \Delta_x AB = \cos \theta ds_e \]

or

\[ \Delta_y AB = \frac{dx}{ds} ds_e \]

where \( \theta \) is the slope angle of straight line AB.

From the relations Eqs. (9) and (10), we have

\[ k_x dx^{\alpha_x} = \frac{dx}{ds} ds^\alpha_k \]

or

\[ dx^{\alpha_x} = k_x \frac{dx}{ds} ds^\alpha_k = \sigma \frac{dx}{ds} ds^\alpha \]

where \( \sigma = k/k_x \). We, therefore, have the following chain rule for fractional calculus

\[ \frac{\partial^u u}{\partial t^\alpha} = \sigma \frac{\partial u}{\partial s} \frac{\partial^s s}{\partial t^\alpha}. \]

Using the following transforms,

\[ s = t^\alpha, \quad (23a) \]

\[ X = x^\beta, \quad (23b) \]

\[ Y = y^\gamma, \quad (23c) \]

\[ Z = z^\delta. \quad (23d) \]

We have

\[ \frac{\partial^u u}{\partial t^\alpha} = \frac{\partial u}{\partial s} \frac{\partial^s s}{\partial t^\alpha} = \sigma \frac{\partial u}{\partial s}, \quad (24a) \]

\[ \frac{\partial^u u}{\partial x^\beta} = \frac{\partial u}{\partial X} \frac{\partial^X X}{\partial x^\beta} = \sigma \frac{\partial u}{\partial X}, \quad (24b) \]

\[ \frac{\partial^u u}{\partial y^\gamma} = \frac{\partial u}{\partial Y} \frac{\partial^Y Y}{\partial y^\gamma} = \sigma \frac{\partial u}{\partial Y}, \quad (24c) \]

\[ \frac{\partial^u u}{\partial z^\delta} = \frac{\partial u}{\partial Z} \frac{\partial^Z Z}{\partial z^\delta} = \sigma \frac{\partial u}{\partial Z}, \quad (24d) \]

where \( \sigma, \sigma_x, \sigma_y, \sigma_z \) are fractal indexes. We can, therefore, easily convert fractional differential equations into partial differential equations.
equations, so that everyone familiar with advanced calculus can deal with fractional calculus without any difficulty.

To determine $\sigma_s$, we consider a special case $s = t^\alpha$ and $u = s^m$, and we have

$$\frac{\partial^\alpha u}{\partial t^\alpha} = \frac{\Gamma(1 + ma) \cdot t^{ma - \alpha}}{\Gamma(1 + ma - \alpha)} = \sigma \frac{\partial u}{\partial s} = \sigma s^m t^{ma - \alpha}.$$  \hfill (25)

We, therefore, can determine $\sigma_s$ as follows

$$\sigma_s = \frac{\Gamma(1 + ma)}{m \Gamma(1 + ma - \alpha)}.$$  \hfill (26)

Other fractal indexes ($\sigma_X, \sigma_Y, \sigma_Z$) can be determined in a similar way.

5. An example

As an example, consider the fractional differential equation

$$\frac{\partial^\alpha u}{\partial t^\alpha} + Bu = 0, \quad 0 < \alpha < 1, \quad u(0) = 1.$$  \hfill (27)

We use this simple example to illustrate how to determine the fractal index. After the transform

$$s = t^\alpha$$  \hfill (28)

we assume that the solution can be expressed in a series in the form

$$u = \sum_{m=0}^{\infty} a_m s^m,$$  \hfill (29)

where $a_m$ ($m = 0, 1, 2, 3, \ldots$) are constants to be further determined.

Submitting Eq. (29) into Eq. (27), we have

$$\frac{\partial}{\partial s} \sum_{m=0}^{\infty} \sigma_s a_m s^m + B \sum_{m=0}^{\infty} a_m s^m = 0$$  \hfill (30)

or

$$\sum_{m=0}^{\infty} m \sigma_s a_m s^{m-1} + B \sum_{m=0}^{\infty} a_m s^m = 0.$$  \hfill (31)

According to Eq. (26), the fractal index $\sigma_s$ can be determined as follows

$$\sigma_s = \frac{\Gamma(1 + ma)}{m \Gamma(1 + ma - \alpha)}.$$  \hfill (32)

From Eqs. (31) and (32), we obtain

$$\frac{\Gamma(1 + ma)}{\Gamma(1 + ma - \alpha)} a_m + B a_{m-1} = 0.$$  \hfill (33)

Generally we begin with $u_0 = u(0) = 1$. After a simple calculation, we have

$$a_m = \frac{(-B)^m}{\Gamma(1 + ma)}.$$  \hfill (34)

We, therefore, obtain the following solution

$$u(s) = \sum_{m=0}^{\infty} \frac{(-B)^m}{\Gamma(1 + ma)} s^m.$$  \hfill (35)

or

$$u(t) = \sum_{m=0}^{\infty} \frac{(-B)^m}{\Gamma(1 + ma)} t^{ma} = E_a(-Bt^\alpha)$$  \hfill (36)

where $E_a$ is a Mittag–Leffler function defined as

$$E_a(t) = \sum_{m=0}^{\infty} \frac{t^m}{\Gamma(1 + ma)}.$$  \hfill (37)

Eq. (36) is the exact solution of the example.

6. Conclusions

Hereby $u$ is assumed to be differentiable with respective to $s, X, Y,$ and $Z$, and the fractional differential equations with Jumarie’s derivative can be easily converted into its classical differential partner by the fractional complex transform, hence everyone familiar with advanced calculus can easily deal with fractional calculus.

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References